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Short Communication

An unusual exact, closed-form solution for axisymmetric vibration of inhomogeneous simply supported circular plates

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Abstract

Apparently for the first time in the literature, an exact closed-form solution is derived for an axisymmetrically vibrating inhomogeneous circular plate that is simply supported at its boundary. The solution is characterized here as an unusual one since for its counterpart—the homogeneous plate, transcendental functions are called for whereas here a solution is found in elementary functions, namely, polynomials. The analysis is based on an inverse vibration problem: Given a candidate mode shape and density distribution, we calculate the plate stiffness so that the governing equation for the plate mode shape is identically satisfied. We also are able to obtain the expression for the corresponding natural frequency.

1. Introduction

The free vibration of circular plates with variable thickness or density has received considerable attention in the literature. Conway [1] found an unusual closed-form solution for a variable-thickness plate on an elastic foundation, in a static setting. Whereas a constant thickness plate involves Kelvin functions, Conway's solution was derived in closed-form. Analogous, closed-form solutions were derived by Harris [2] for plates of variable thickness, free on their boundary and by Lenox and Conway [3] who studied plates with arbitrary conditions, and with parabolic thickness

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variation. In recent papers, Elishakoff [4,5] derived closed-form solutions for inhomogeneous circular plates that are either clamped, or free at the boundary. Here we derive closed-form solutions for the inhomogeneous plates that are simply supported at their boundary.

There appears to be a single monograph [6] solely devoted to plates of variable thickness. There are several papers dedicated to vibrations of plates with thickness variations.

Axisymmetric vibration of circular plates of linearly varying thickness was studied by Prasad et al. [7], whereas plates with double linear thickness were studied by Sing and Sascena [8]. Various analytical and approximate techniques have been studied [9–17].

In this paper, we find closed form solutions to an inverse vibration problem for a simply supported circular plate. Given a candidate mode shape and density distribution, we calculate the plate stiffness so that the governing equation and boundary conditions for the plate mode shape are identically satisfied. We also are able to obtain the expression for the corresponding natural frequency. The solution is characterized here as an unusual one since for its counterpart—the homogeneous plate, transcendental functions are called for whereas here a solution is found in elementary functions, namely, polynomials. The solutions can also serve as benchmarks for validations of numerical techniques.

2. Basic equations

The differential equation that governs the free axisymmetric vibration of the circular plate with variable thickness reads [4]

$$D(r)r^3\nabla^4 W + \frac{\mathrm{d}D}{\mathrm{d}r}\left(2r^3\frac{\mathrm{d}^3 W}{\mathrm{d}r^3} + (2+v)r^2\frac{\mathrm{d}^2 W}{\mathrm{d}r^2} - r\frac{\mathrm{d}W}{\mathrm{d}r}\right) + \frac{\mathrm{d}^2 D}{\mathrm{d}r^2}\left(r^3\frac{\mathrm{d}^2 W}{\mathrm{d}r^2} + vr^2\frac{\mathrm{d}W}{\mathrm{d}r}\right) - \rho h\omega^2 r^3 W = 0, \tag{1}$$

where $D(r)$ is the plate flexural rigidity $D = Eh^3/12(1-v^2)$, $h(r)$ is the thickness, v the Poisson ratio, E the Young's modulus, and ρ is the mass density. Here ∇^2 is the Laplacian operator in polar coordinates ($\nabla^2 = \mathrm{d}^2/\mathrm{d}r^2 + r^{-1}\mathrm{d}/\mathrm{d}r$) where r denotes the radial coordinate and W the mode shape. Defining the inertial term (mass per unit area) $\delta(r) = \rho h$, we pose to find the stiffness distribution $D(r)$ and natural frequency ω such that Eq. (1) is identically satisfied for a specified $\delta(r)$ and $W(r)$.

To obtain a candidate mode shape, consider the static displacement of a *uniform* circular simply supported plate under uniform load q_0 per unit area. From Ref. [18], we have

$$w = \frac{q_0}{64D}(R^2 - r^2)(\theta R^2 - r^2),$$

where R is the radius of the plate and the parameter θ depends solely of the Poisson ratio.

$$\theta = (5+v)/(1+v).$$

Thus we seek to determine under what conditions is the function

$$W = (R^2 - r^2)(\theta R^2 - r^2) \tag{2}$$

a solution to Eq. (1). If we assume that the inertial term is a specified polynomial of degree m,

$$\delta = \sum_{i=0}^{m} a_i r^i, \tag{3}$$

then it follows from Eq. (1) and the observation that W is a fourth degree polynomial, that the stiffness must be of degree $m+4$:

$$D = \sum_{i=0}^{m+4} b_i r^i. \tag{4}$$

In the following sections we find the coefficients b_i and natural frequency ω for the case of a constant, linear and parabolic inertial term.

3. Constant inertial term ($m=0$)

We are given $\delta = a_0 > 0$, and the stiffness is sought as a fourth-order polynomial

$$D(r) = b_0 + b_1 r + b_2 r^2 + b_3 r^3 + b_4 r^4.$$

Inserting this expression along with Eq. (2) into the differential equation (1), we obtain

$$\sum_{i=0}^{7} c_i r^i = 0,$$

where

$$\begin{aligned}
&c_0 = 0, \quad c_1 = 0, \quad c_2 = -4(3+v)R^2 b_1, \quad c_3 = 64 b_0 - 16(3+v)R^2 b_2 - a_0\omega^2 R^4 \frac{5+v}{1+v},\\
&c_4 = 12(11+v)b_1 - 36(3+v)R^2 b_3,\\
&c_5 = 32(7+v)b_2 - 64(3+v)R^2 b_4 + 2a_0\omega^2 R^2 \frac{3+v}{1+v},\\
&c_6 = (340+60v)b_3, \quad c_7 = 96(5+v)b_4 - a_0\omega^2.
\end{aligned} \tag{5}$$

Demanding that the above set of (non-trivial) coefficient vanish, yields a set of six homogeneous linear equations on the six unknowns $\{b_0, b_1, \ldots, b_4, \omega^2\}$. Fortunately, the determinant of the associated coefficient matrix is zero, hence a non-trivial solution is obtainable. Setting the coefficient c_7 to zero, we obtain the natural frequency

$$\omega^2 = 96(5+v)b_4/a_0, \tag{6}$$

where b_4 is arbitrary but positive.

Upon substitution of Eq. (6) into Eq. (5), the remaining equations yield the coefficients in the stiffness:

$$b_0 = \frac{57+18v+v^2}{2(1+v)} R^4 b_4, \quad b_1 = 0, \quad b_2 = -4\frac{3+v}{1+v} R^2 b_4, \quad b_3 = 0.$$

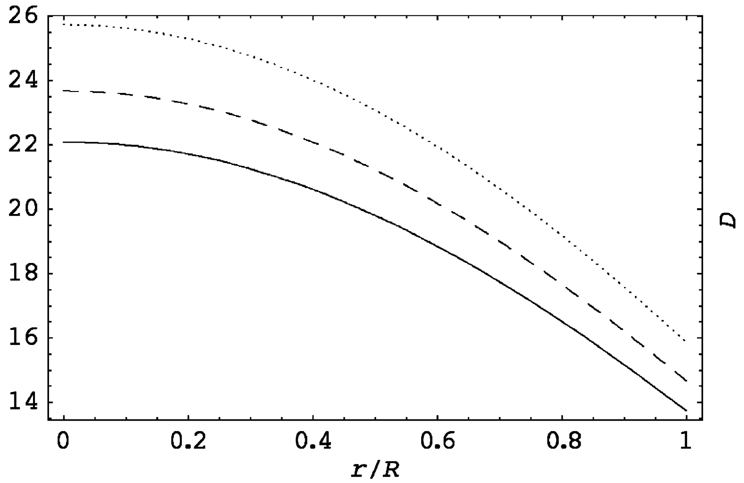

Fig. 1. Stiffness distribution of the simply supported circular plate with constant inertial term for several values of the Poisson ratio: ——, $v = \frac{1}{2}$; - - - -, $v = \frac{1}{3}$; ⋯⋯, $v = \frac{1}{6}$.

Hence, the stiffness reads

$$D(r) = \left(\frac{57 + 18v + v^2}{2(1+v)} R^4 - 4\,\frac{3+v}{1+v} R^2 r^2 + r^4 \right) b_4.$$

Fig. 1 depicts the stiffness for various values of the Poisson ratio v.

4. Linearly varying inertial term ($m = 1$)

We are given $\delta = a_0 + a_1 r$, and the stiffness is sought as a fifth-order polynomial

$$D(r) = b_0 + b_1 r + b_2 r^2 + b_3 r^3 + b_4 r^4 + b_5 r^5.$$

Instead of set (5), we get here seven linear algebraic equations in the seven unknowns $\{b_0, b_1, \ldots, b_5, \omega^2\}$:

$$\begin{aligned}
& -4(3+v)R^2 b_1 = 0, \\
& 64b_0 - 16(3+v)R^2 b_2 - a_0\omega^2 R^4\,\frac{5+v}{1+v} = 0, \\
& 12(11+v)b_1 - 36(3+v)R^2 b_3 - a_1\omega^2 R^4\,\frac{5+v}{1+v} = 0, \\
& 32(7+v)b_2 - 64(3+v)R^2 b_4 + 2a_0\omega^2 R^2\,\frac{3+v}{1+v} = 0, \\
& (340+60v)b_3 - 100(3+v)R^2 b_5 + 2a_1\omega^2 R^2\,\frac{3+v}{1+v} = 0, \\
& 96(5+v)b_4 - a_0\omega^2 = 0, \\
& (644+140v)b_5 - a_1\omega^2 = 0.
\end{aligned} \tag{7}$$

In order to have a non-trivial solution, the determinant of the coefficient matrix must vanish

$$R^6 \frac{(3+v)(7+v)(5+v)(5546+2619v+364v^2+15v^3)}{1+v} a_1 = 0.$$

Thus $a_1 = 0$, and we obtain the solution in the previous section.

5. Parabolically varying inertial term ($m = 2$)

For $m = 2$, i.e. the plate whose material density varies parabolically,

$$\delta(r) = a_0 + a_1 r + a_2 r^2, \tag{8}$$

the bending stiffness has to be a sixth-order polynomial

$$D(r) = b_0 + b_1 r + b_2 r^2 + b_3 r^3 + b_4 r^4 + b_5 r^5 + b_6 r^6. \tag{9}$$

Substitution of Eq. (2) in conjunction with Eqs. (8) and (9) into the governing differential equation (1) yields

$$\sum_{i=0}^{9} d_i r^i = 0,$$

where

$$\begin{aligned}
&d_0 = 0, \quad d_1 = 0, \quad d_2 = -4R^2(3+v)b_1,\\
&d_3 = 64b_0 - 16R^2(3+v)b_2 - a_0\omega^2 R^4 \frac{5+v}{1+v},\\
&d_4 = 12(11+v)b_1 - 36R^2(3+v)b_3 - a_1\omega^2 R^4 \frac{5+v}{1+v},\\
&d_5 = 32(7+v)b_2 - 64R^2(3+v)b_4 + \omega^2\left(2a_0R^2 \frac{3+v}{1+v} - a_2R^4 \frac{5+v}{1+v}\right),\\
&d_6 = 20(17+3v)b_3 - 100R^2(3+v)b_5 + 2a_1\omega^2 R^2 \frac{3+v}{1+v},\\
&d_7 = 96(5+v)b_4 - 144R^2(3+v)b_6 - \omega^2\left(a_0 - 2a_2R^2 \frac{3+v}{1+v}\right),\\
&d_8 = (644+140v)b_5 - a_1\omega^2, \quad d_9 = (832+192v)b_6 - a_2\omega^2.
\end{aligned} \tag{10}$$

As in the case of the constant inertial term, we demand that all $d = 0$, thus, we get a set of eight equations with eight unknowns (seven coefficients b, and ω^2). The resulting determinantal equation is

$$(3+v)(7+v)a_1(360\,490 + 325\,523v + 113\,630v^2 + 19\,024v^3 + 1512v^4 + 45v^5)/(1+v) = 0.$$

In order for the homogeneous system to have a non-trivial solution we must demand the coefficient a_1 to vanish. Substituting $a_1 = 0$ into set (10) and using the last equation for the determination of the frequency, we obtain

$$\omega^2 = 64(13 + 3v)b_6/a_2 \tag{11}$$

and the stiffness coefficients

$$\begin{aligned} b_0 &= R^4 b_6[(3285 + 2670v + 744v^2 + 82v^3 + 3v^4)R^2 a_2 \\ &\quad + (20\,748 + 14\,304v + 3496v^2 + 352v^3 + 12v^4)a_0]/12(35 + 47v + 13v^2 + v^3)a_2, \\ b_1 &= 0, \\ b_2 &= \frac{(1095 + 525v + 73v^2 + 3v^3)R^2 a_2 - (2184 + 1544v + 344v^2 + 24v^3)a_0}{3(35 + 47v + 13v^2 + v^3)a_2} R^2 b_6, \\ b_3 &= 0, \quad b_4 = -\frac{(285 + 140v + 15v^2)R^2 a_2 - (52 + 64v + 12v^2)a_0}{6(5 + 6v + v^2)a_2} b_6, \quad b_5 = 0, \end{aligned}$$

where b_6 is an arbitrary constant. It follows from Eq. (11) that the ratio b_6/a_2 must be positive. We have two sub-cases: (1) both b_6 and a_2 are positive, or (2) both are negative. In the former case (1) the necessary condition for positiveness of the stiffness $b_0 \geqslant 0$ is identically satisfied. In the latter case (2) the above inequality reduces to

$$\begin{aligned} &(3285 + 2670v + 744v^2 + 82v^3 + 3v^4)R^2 \\ &\quad + (20\,748 + 14\,304v + 3496v^2 + 352v^3 + 12v^4)a_0/a_2 \leqslant 0, \end{aligned}$$

leading to the inequality

$$\frac{a_0}{|a_2|R^2} \geqslant \frac{3285 + 2670v + 744v^2 + 82v^3 + 3v^4}{20\,748 + 14\,304v + 3496v^2 + 352v^3 + 12v^4}. \tag{12}$$

It is interesting to note that in this case the requirement that $\delta(R)$ be non-negative implies

$$\frac{a_0}{|a_2|R^2} \geqslant 1,$$

which is stronger than inequality (12)

As an example, for $v = \frac{1}{3}$, the stiffness reads

$$\begin{aligned} D(r) = R^6 \bigg[&\left(\frac{3595}{528} + \frac{497a_0}{12R^2 a_2} \right) + \left(\frac{719}{88} - \frac{35a_0}{2R^2 a_2} \right) \left(\frac{r}{R} \right)^2 \\ &+ \left(-\frac{125}{16} + \frac{7a_0}{4R^2 a_2} \right) \left(\frac{r}{R} \right)^4 + \left(\frac{r}{R} \right)^6 \bigg] b_6. \end{aligned}$$

Fig. 2 depicts the variation of $D(r)$ for different values of $t \equiv a_0/(R^2 a_2)$.

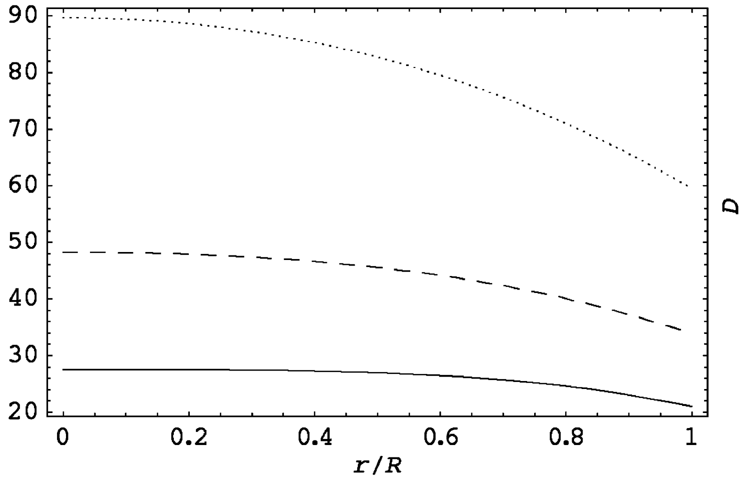

Fig. 2. Stiffness distribution of the simply supported circular plate with parabolic inertial term and $v = \frac{1}{3}$ for different values of t: ——, $t = \frac{1}{2}$; - - - -, $t = 1$; ⋯⋯, $t = 2$.

6. Cubic inertial term ($m = 3$)

Proceeding as before, the following set of nine linear algebraic equations in the nine unknowns $\{b_0, b_1, \ldots, b_7, \omega^2\}$ is obtained

$$-4R^2(3+v)b_1 = 0, \quad 64b_0 - 16R^2(3+v)b_2 - a_0\omega^2 R^4 \frac{5+v}{1+v} = 0, \tag{13}$$

$$12(11+v)b_1 - 36R^2(3+v)b_3 - a_1\omega^2 R^4 \frac{5+v}{1+v} = 0,$$

$$32(7+v)b_2 - 64R^2(3+v)b_4 + \omega^2\left(2a_0R^2\frac{3+v}{1+v} - a_2R^4\frac{5+v}{1+v}\right) = 0,$$

$$20(17+3v)b_3 - 100R^2(3+v)b_5 + \omega^2\left(2a_1R^2\frac{3+v}{1+v} - a_3R^4\frac{5+v}{1+v}\right) = 0,$$

$$96(5+v)b_4 - 144R^2(3+v)b_6 - \omega^2\left(a_0 - 2a_2R^2\frac{3+v}{1+v}\right) = 0,$$

$$(644+140v)b_5 - 196(3+v)R^2b_7 - \omega^2\left(a_1 - 2a_3R^2\frac{3+v}{1+v}\right) = 0, \tag{14}$$

$$(832+192v)b_6 - a_2\omega^2 = 0, \quad (1044+252v)b_7 - a_3\omega^2 = 0. \tag{15}$$

The determinantal equation stemming from it reads

$$\begin{aligned}&(2\,527\,200 + 3\,153\,105v + 1\,582\,173v^2 + 406\,618v^3 + 56\,058v^4\\&\quad + 3893v^5 + 105v^6)R^2a_3 + (10\,454\,210 + 11\,963\,597v + 5\,573\,931v^2\\&\quad + 1\,347\,106v^3 + 177\,016v^4 + 11\,889v^5 + 315v^6)a_1 = 0.\end{aligned}$$

Thus the coefficients a_1 and a_3 in the density distribution must satisfy the relation

$$a_1 = -\frac{38\,880 + 31\,761v + 8865v^2 + 971v^3 + 35v^4}{160\,834 + 114\,773v + 28\,889v^2 + 2983v^3 + 105v^4} R^2 a_3.$$

Substituting this equation into Eqs. (13)–(15), we find

$$\omega^2 = 36(29 + 7v)b_7/a_3 \tag{16}$$

and obtain the following solution for the coefficients in the stiffness:

$$b_0 = 3[(95\,265 + 100\,425v + 40\,266v^2 + 7586v^3 + 661v^4 + 21v^5)R^2 a_2 + (601\,692 + 560\,052v + 201\,512v^2 + 34\,680v^3 + 2812v^4 + 84v^5)a_0] \quad R^4 b_7/64a_3 M,$$

$$b_1 = 0,$$

$$b_2 = 3[(31\,755 + 22\,890v + 5792v^2 + 598v^3 + 21v^4)R^2 a_2 - (63\,336 + 60\,064v + 20\,784v^2 + 3104v^3 + 168v^4)a_0]R^2 b_7/16a_3 M,$$

$$b_3 = (64\,800 + 44\,295v + 10\,597v^2 + 1041v^3 + 35v^4)R^4 b_7/N,$$

$$b_4 = 3b_7[(1508 + 2220v + 796v^2 + 84v^3)a_0 - (8265 + 6055v + 1415v^2 + 105v^3)R^2 a_2]/32a_3(65 + 93v + 31v^2 + 3v^3),$$

$$b_5 = -2R^2(25\,527 + 20\,092v + 5424v^2 + 584v^3 + 21v^4)b_7/N,$$
$$b_6 = 9(29 + 7v)a_2 b_7/16a_3(13 + 3v).$$

where

$$M = 455 + 716v + 310v^2 + 52v^3 + 3v^4, \; N = 5546 + 8165v + 2983v^2 + 379v^3 + 15v^4.$$

and b_7 an arbitrary constant with the same sign as a_3 (see Eq. (16)). For the particular case $v = \frac{1}{3}$, the stiffness equals

$$D(r) = R^6 \left[\frac{168\,965a_2}{19\,712a_3} + \frac{3337a_0}{64R^2 a_3} + \left(\frac{101\,379a_2}{9856a_3} - \frac{705a_0}{32R^2 a_3} \right) \left(\frac{r}{R} \right)^2 + \frac{21\,524R}{2295} \left(\frac{r}{R} \right)^3 + \left(-\frac{17\,625a_2}{1792a_3} + \frac{141a_0}{64R^2 a_3} \right) \left(\frac{r}{R} \right)^4 - \frac{389R}{51} \left(\frac{r}{R} \right)^5 + \frac{141a_2}{112a_3} \left(\frac{r}{R} \right)^6 + R \left(\frac{r}{R} \right)^7 \right] b_7.$$

7. General inertial term (m$\geqslant$4)

Consider now the general expression of the inertial term given in Eq. (3) and the stiffness in Eq. (4), for $m \geqslant 4$. Substitution of Eqs. (2)–(4) into the terms of the differential equation (1) yields

$$r^3 D(r) \nabla^4 W = 64r^3 \sum_{i=0}^{m+4} b_i r^i, \tag{17}$$

$$\frac{\mathrm{d}D}{\mathrm{d}r}\left(2r^3\frac{\mathrm{d}^3W}{\mathrm{d}r^3}+(2+v)r^2\frac{\mathrm{d}^2W}{\mathrm{d}r^2}-r\frac{\mathrm{d}W}{\mathrm{d}r}\right)=4[(17+3v)r^2-(3+v)R^2]r^2\sum_{i=1}^{m+4}ib_ir^{i-1}, \tag{18}$$

$$\frac{\mathrm{d}^2D}{\mathrm{d}r^2}\left(r^3\frac{\mathrm{d}^2W}{\mathrm{d}r^2}+vr^2\frac{\mathrm{d}W}{\mathrm{d}r}\right)=4(3+v)(r^2-R^2)r^3\sum_{i=2}^{m+4}i(i-1)b_ir^{i-2}, \tag{19}$$

$$-\rho h\omega^2r^3W=-\omega^2r^3(R^2-r^2)(\theta R^2-r^2)\sum_{i=0}^{m}a_ir^i. \tag{20}$$

Demanding the sum of Eqs. (17)–(20) to be zero, we obtain

$$\sum_{i=2}^{m+7}g_ir^i=0,$$

where the coefficients g_i are

$$g_2=-4(3+v)b_1,$$
$$g_3=64b_0-16R^2(3+v)b_2-\theta a_0R^4\omega^2,$$
$$g_4=12(11+v)b_1-36R^2(3+v)b_3-\theta a_1R^4\omega^2,$$
$$g_5=32(7+v)b_2-64R^2(3+v)b_4-\omega^2[\theta a_2R^4-(1+\theta)a_0R^2],$$
$$g_6=20(17+3v)b_3-100R^2(3+v)b_5-\omega^2[\theta a_3R^4-(1+\theta)a_1R^2],$$

and for $7\leqslant i\leqslant m+3$

$$g_i=4(i-1)[(v+3)(i-1)+2(1-v)]b_{i-3}-4(i-1)^2R^2(3+v)b_{i-1}-\omega^2[\theta R^4a_{i-3}-(1+\theta)R^2a_{i-5}+a_{i-7}],$$

$$g_{m+4}=4(m+3)[v+m(v+3)+11]b_{m+1}-4R^2(m+3)^2(3+v)b_{m+3}-\omega^2[a_{m-3}-(1+\theta)R^2a_{m-1}],$$

$$g_{m+5}=4(m+4)[m(v+3)+2(v+7)]b_{m+2}-4R^2(m+4)^2(3+v)b_{m+4}-\omega^2[a_{m-2}-(1+\theta)R^2a_m],$$

$$g_{m+6}=4(m+5)[3v+17+m(v+3)]b_{m+3}-\omega^2a_{m-1},$$
$$g_{m+7}=4(m+6)[m(v+3)+4(v+5)]b_{m+4}-\omega^2a_m.$$

We demand all coefficients g_i to be zero, thus, we get a set of $m+6$ homogeneous linear algebraic equations for the $m+6$ unknowns $\{b_0,b_1,\ldots,b_{m+4},\omega^2\}$. In order to find a non-trivial solution the determinant of the associated coefficient matrix must vanish. We expand the determinant along the last column of the matrix of the set, getting a linear algebraic expression with a_i as coefficients. The determinantal equation yields a condition for which the non-trivial solution is obtainable. In this case the general expression of the natural frequency squared is obtained from the equation

$g_{m+7} = 0$, resulting in

$$\omega^2 = \frac{4(m+6)[m(v+3)+4(v+5)]}{a_m}\, b_{m+4}. \tag{21}$$

Note that the formulas pertaining to the cases $m = 0,\ 2$ and 3 are formally obtainable from Eq. (21) by appropriate substitution.

8. An alternative mode shape

In the previous sections, we were able to obtain the stiffness distribution corresponding to a polynomial density such that the fourth degree polynomial given in Eq. (2) served as a mode shape for the simply supported plate. The question naturally arises: can we find a lower order polynomial mode shape? Since there are two boundary conditions that need to be satisfied, we try the second degree polynomial

$$W = (r-R)[vr - R(v+2)], \tag{22}$$

which satisfies the simply supported boundary conditions $W = M_r = 0$ on $r = R$ (recall that $M_r = -D(\partial^2 W/\partial r^2 + v/r\partial W/\partial r)$). In connection with this mode shape, we will limit our discussion to the case of a constant density. Thus given $\delta = a_0 > 0$, we want to find $D(r)$ and the natural frequency ω such that Eq. (22) is a solution of Eq. (1). It is sufficient to consider fourth degree polynomials:

$$D(r) = b_0 + b_1 r + b_2 r^2 + b_3 r^3 + b_4 r^4.$$

Inserting this expression along with Eq. (22) into the differential equation, we find that the 6 unknowns $\{b_0, b_1, \ldots, b_4, \omega^2\}$ must satisfy the following system of 5 homogeneous linear equations:

$$\begin{aligned}
&b_0 = 0, \quad vb_1 + R(1-2v)b_2 = 0,\\
&8v(v+1)b_2 - 4R(v+1)(3v-1)b_3 - R^2 a_0 (v+2)\omega^2 = 0,\\
&9vb_3 + 3R(1-4v)b_4 + Ra_0\omega^2 = 0, \quad 32(v+1)b_4 - a_0\omega^2 = 0.
\end{aligned}$$

The last equation yields the natural frequency

$$\omega^2 = 32(v+1)b_4/a_0, \tag{23}$$

where b_4 is arbitrary but positive. The remaining stiffness coefficients are then given by

$$\begin{aligned}
&b_0 = 0, \quad b_1 = R^3(2v-1)(12v^2+59v+35)b_4/18v^3,\\
&b_2 = R^2(12v^2+59v+35)b_4/18v^3, \quad b_3 = -5R(4v+7)b_4/9v.
\end{aligned}$$

Since $b_0 = 0$, we must have $b_1 \geqslant 0$ in order that the stiffness shall be non-negative. Based upon the above solution, this implies that we must have $v \geqslant \frac{1}{2}$. The plate stiffness is thus given by

$$D(r) = \left[\xi^4 - \frac{5(4v+7)}{9v}\,\xi^3 + \frac{12v^2+59v+35}{18v^2}\,\xi^2 + \frac{(2v-1)(12v^2+59v+35)}{18v^3}\,\xi\right] R^4 b_4,$$

where $\xi = r/R$. Fig. 3 shows the stiffness variation over the plate for $v = \frac{1}{2}$ and $\frac{2}{3}$.

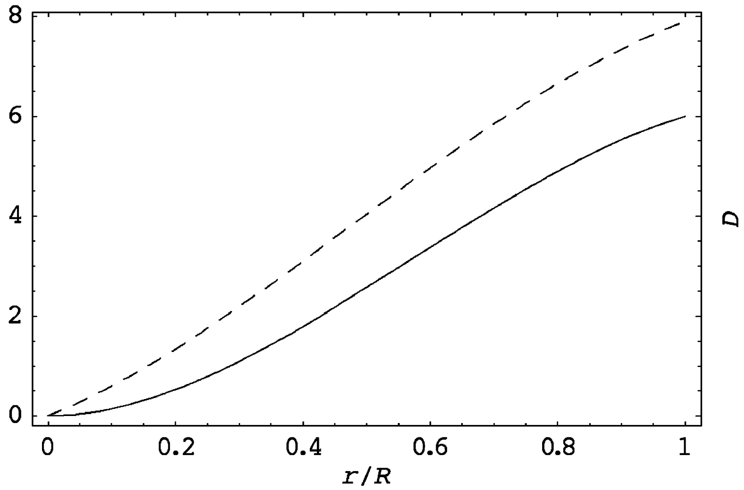

Fig. 3. Stiffness distribution of the simply supported plate for quadratic mode shape: ——, $\nu = \frac{1}{2}$; - - - -, $\nu = \frac{2}{3}$.

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